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ARTICLE

Invariants in Birational Mapping: An In-Depth and Comprehensive Guide to Their Properties, Applications, and Mathematical Significance

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Abstract
This research article provides a thorough exploration of invariants in birational mapping, focusing on their properties, applications, and mathematical significance in algebraic geometry. Birational mappings play a crucial role in connecting algebraic varieties through rational transformations while preserving fundamental geometric properties. The study of invariants under birational mappings elucidates essential aspects of variety structure and classification, offering insights into singularities and geometric invariance. The article covers foundational concepts of birational invariants, their computational methods, and their applications in resolving geometric problems and conjectures. Through detailed examples and discussions of recent theoretical developments, this guide aims to equip mathematicians, researchers, and students with a deep understanding of birational invariants' theoretical framework and practical utility. This work contributes to advancing the understanding of algebraic geometry and its applications across various mathematical disciplines. .

Keywords: Algebraic geometry; Birational mapping; Computational methods; Geometric invariance; Invariants; Mathematical significance; Rational transformations; Variety structure

Abbreviations: MMP: Minimal Model Program

1. Introduction

In the realm of linear algebra and group theory, invariants play a crucial role in understanding the observable behaviour of imperative programs and their transformations. These mathematical concepts provide a robust framework for analyzing and manipulating complex structures, making them indispensable tools for researchers and practitioners alike [1, 2, 3, 4]. This comprehensive guide delves into the intricate world of invariants in birational maps, exploring their significance, applications, and the theoretical underpinnings that govern their behavior. We will unravel the fundamental principles underlying these mathematical constructs, shedding light on their utility in various domains, from abstract algebra to computational geometry [5, 6, 7, 8].

2. Fundamental Concepts of Invariants

Birational maps are rational maps between algebraic varieties that have an inverse rational map. Varieties that are birational are said to be birationally equivalent. A variety is rational if it is birational to affine or projective space. Examples of rational varieties include the circle

$$x^2 + y^2 - 1 = 0$$

and smooth quadric hypersurfaces of any dimension [9, 10, 11, 12, 13, 14].

Birational invariants are properties of varieties that are preserved under birational equivalence, such as:

- **Plurigenera:** The dimensions of the space of global n-forms of weight m,

$$Pm(X) = h0(X, OX(mKX))$$

.

- **Kodaira dimension:** The Iitaka dimension of the canonical divisor,

$$\varkappa(X) = \varkappa(X, KX)$$

.

- **Fundamental group:** The fundamental group of smooth projective varieties.

In higher dimensions, the minimal model program aims to classify varieties up to birational equivalence by finding a unique minimal model in each birational class. The birational automorphism group can also be a birational invariant, with rational varieties having very large automorphism groups compared to varieties of general type [15, 16, 17]. Other important concepts related to birational invariants include:

- **Iitaka dimension:** The maximum dimension of the image of X under the linear systems $|mkD|$, denoted as $\kappa(X, D)$ for a Q-divisor D on a normal projective variety X.
- **Irregularity:** The dimension of the space of 1-forms,

$$q(X) = h0(X, \Omega^1_X)$$

If X and Y are smooth projective varieties that are birational, then they have the same invariants defined above.

3. Properties of Birational Invariants

Invariants are fundamental mathematical concepts that capture the intrinsic properties of structures and objects, remaining unchanged under certain transformations or operations. They provide a stable reference point, enabling the identification and comparison of complex structures despite undergoing transformations. The study of invariants is crucial as it helps uncover properties that remain invariant, leading to a deeper understanding of mathematical structures and their transformations [18, 19, 20, 21].

- **Formal Definitions:** Invariants can be formally defined in three ways:
  1. Properties that remain unchanged under group action.
  2. Properties independent of the presentation or representation of an object.
  3. Properties that remain unchanged under perturbations or small changes.
- **Examples:** Some examples of invariants include:
  - The number of objects in a finite set
  - Angles and ratios of distances

- The degree of a polynomial
- Euclidean distance
- The determinant, trace, eigenvectors, and eigenvalues of a linear endomorphism
- Invariant sets, such as a circle being an invariant set under rotation about its center

Invariants play a vital role in various areas of mathematics and its applications [22]:

- **Algebra:** In group theory, invariants are essential for studying group actions, with applications in coding theory and cryptography.
- **Geometry:** Invariants are fundamental in transformation geometry, such as the use of cross-ratios in projective geometry.
- **Physics:** The concept of invariants is crucial in the formulation of physical laws, like the spacetime interval in special relativity.

4. Computational Approaches to Invariant Calculation

Birational invariants encompass a wide range of properties that remain unchanged under birational equivalence, shedding light on the fundamental characteristics of algebraic varieties. One notable invariant is the fundamental group, which plays a crucial role in understanding the topology of these structures (Fig. 1) [23, 24, 25, 26, 27].

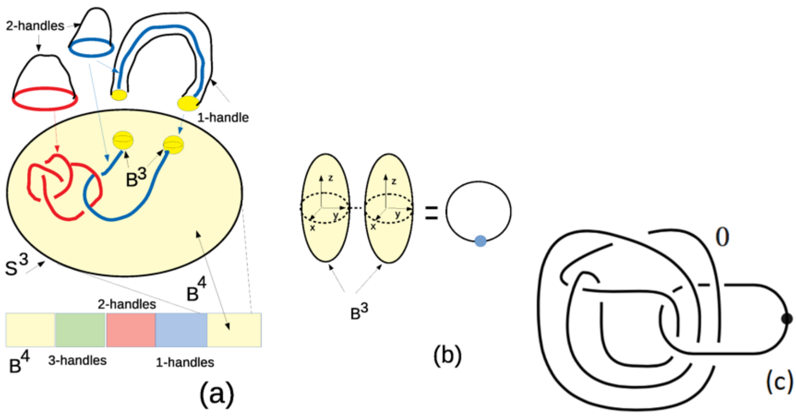

Figure 1. Handlebody of a 4-manifold.

- For smooth projective surfaces, the fundamental group is a birational invariant. Specifically, blowing up a smooth point on a surface does not alter its fundamental group [8]. This property ensures that the topological structure remains intact under certain transformations.
- Surfaces in P^3 and, more generally, sets $X \Subset P^n$ defined by a single homogeneous equation of degree > 0, exhibit a remarkable property: the pair (P^n, X) is at least $(n-1)$-connected. This connectivity condition highlights the intricate relationship between the ambient projective space and the algebraic variety itself.

While the fundamental group captures topological invariance, there are other birational invariants that are not homotopy invariants. These include [28, 29, 30, 31]:

- The dimensions of spaces of p-forms

- The spaces of sections of tensor powers of the canonical bundle (plurigenera)
- The quantity $K^2 + \rho$, where K is the self-intersection of the canonical divisor class, and ρ is the rank of the Néron-Severi group

These invariants provide insights into the geometric and algebraic properties of the varieties, offering a rich tapestry of information for further exploration.

For surfaces with a single singular point x, the fundamental group of S-x (the surface with the singular point removed) can be related to the formal isomorphism class of the singularity. However, the precise details of this relationship are not yet fully understood, leaving room for further investigation and discovery [32].

5. Applications of Invariants in Algebraic Geometry

The study of birational invariants has led to several remarkable discoveries and conjectures, shedding light on the intricate relationships between algebraic varieties and their transformations. Here are some notable observations and conjectures [33, 34]:

- For a smooth projective surface X with at worst Du Val singularities, if X has 16 singular points, then the fundamental group of the smooth part X_0 is infinite, even though X itself is simply connected. This intriguing result highlights the potential discrepancy between the topological properties of a variety and its singular counterpart.
- There is a conjecture that for a Q-Fano n-fold V, the topological fundamental group of the smooth part V_0 is finite. This conjecture, if proven, would provide valuable insights into the behavior of these specific algebraic varieties under birational transformations.
- The paper [9] constructs invariants of birational maps with values in the Kontsevich-Tschinkel group and in the truncated Grothendieck groups of varieties. These invariants are morphisms of groupoids and are well-suited for investigating the structure of the Grothendieck ring and L-equivalence. By building on known constructions of L-equivalence, the paper proves new unexpected results about Cremona groups, further expanding our understanding of birational transformations.
- In the context of bimeromorphic mappings on compact Kähler surfaces, the paper [10] introduces several conditions and concepts:
  1. The condition (1) separates the obstructions to forward and backward dynamics, ensuring that the obstructions do not overlap.
  2. If the spectral radius ρ of f^*on $H^{1,1}(X)$ is greater than 1, there exist stable/unstable currents μ whose cohomology classes generate the $f^*$and f^* eigenspaces for ρ.
  3. A stronger condition (3) is introduced, which is equivalent to the local potentials g^+being bounded below. This condition has implications for the dynamics of the iterates f^n.

These observations and conjectures contribute to our understanding of birational invariants and their role in unraveling the behavior of algebraic varieties under transformations, opening new avenues for exploration and discovery [35, 36, 37, 38].

6. Geometric Significance of Invariants

The study of birational invariants has revealed several remarkable properties and relationships that shed light on the intricate nature of algebraic varieties and their transformations. Here are some key observations (Fig. 2):

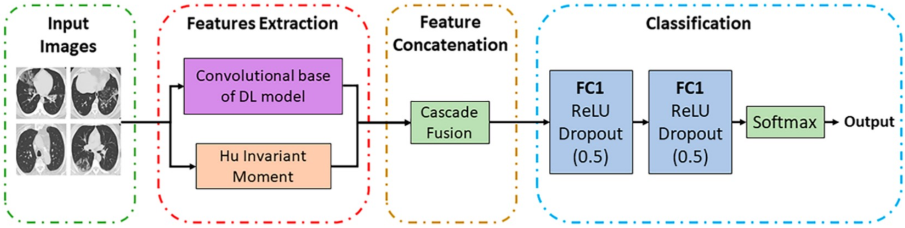

Figure 2. Block diagram of the proposed method.

Blowing up a smooth point on a surface does not change the fundamental group, making the fundamental group of a smooth projective surface a birational invariant [39]. This property ensures that the topological structure remains intact under certain transformations [40].

- Surfaces in P^3 are simply connected, and more generally, for a set XP^n defined by a single homogeneous equation of degree > 0, the pair (P^n, X) is at least (n-1)-connected. This connectivity condition highlights the intricate relationship between the ambient projective space and the algebraic variety itself.
- While the fundamental group captures topological invariance, there are other birational invariants that are not homotopy invariants, such as:
  - The dimensions of spaces of p-forms
  - The spaces of sections of tensor powers of the canonical bundle (plurigenera)
  - The quantity K^2+, where K is the self-intersection of the canonical divisor class, and is the rank of the Néron-Severi group [41, 42]

These invariants provide insights into the geometric and algebraic properties of the varieties, offering a rich tapestry of information for further exploration. For surfaces with a single singular point x, the fundamental group of S-x (the surface with the singular point removed) can be related to the formal isomorphism class of the singularity. Notably, cones over smooth plane curves are simply connected [43].

7. Classification and Resolution of Singularities

Birational invariants play a pivotal role in the study of birational geometry, providing a framework for classifying and understanding the structure of algebraic varieties up to birational equivalence. These invariants are properties that remain unchanged under birational transformations, offering a stable reference point for exploring the intricate relationships between different varieties [44].

A fundamental example of a birational invariant is the Riemann surface and its geometric genus, as demonstrated by Riemann in his thesis. Birational equivalent algebraic curves yield the same Riemann surface, highlighting the invariance of this property under such transformations (Fig. 3).

Another notable instance is the Hodge numbers $h^{0,1}$ and $h^{0,2}$ of a non-singular projective complex surface. These numbers are birational invariants, meaning they remain constant under birational equivalence. However, it's important to note that the Hodge number $h^{1,1}$is not a birational invariant, as it can be altered by blowing up a point on the surface [45].

Other significant birational invariants include:

- Dimensions of spaces of p-forms

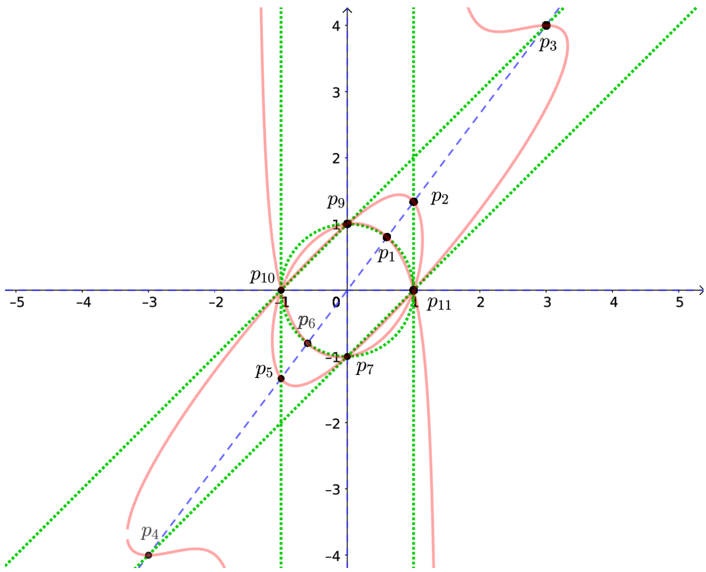

Figure 3. Invariant curves of the sextic pencil.

- Spaces of sections of tensor powers of the canonical bundle (plurigenera)
- The torsion subgroup T(X) of $H^3(X, Z)$ for a smooth projective variety X
- For surfaces, the quantity $K^2 + \rho$, where K is the canonical divisor class and ρ is the rank of the Néron-Severi group

Interestingly, while surfaces in P^3 are simply connected, and more generally, for a set $X \subset P^n$ defined by a single homogeneous equation of degree > 0, the pair (P^n, X) is at least (n-1)-connected, the fundamental group of the smooth part X_0 can be quite intricate for a smooth projective surface X with Du Val singularities, even if X itself is simply connected [46].

8. Recent Advances in Birational Invariants

Birational invariants have profound implications in the realm of algebraic geometry, particularly in the classification and study of algebraic varieties. One notable application lies in the Minimal Model Program (MMP), a powerful tool for understanding the birational geometry of algebraic varieties [47, 48].

The MMP aims to construct a unique minimal model within each birational equivalence class of varieties. This minimal model serves as a representative for the entire class, capturing the essential properties and characteristics. The program relies heavily on birational invariants to guide the transformations and identify the minimal models [49].

- **Canonical Divisors:** The canonical divisor, denoted by K_X, plays a central role in the MMP. Its self-intersection number, K_X^2, is a birational invariant, providing insights into the geometric properties of the variety.
- **Kodaira Dimension:** The Kodaira dimension, denoted by $\kappa(X)$, is a birational invariant that measures the growth rate of the plurigenera of a variety. It serves as a crucial indicator for determining the appropriate steps in the MMP.
- **Mori Cone:** The Mori cone, a convex cone in the $N^1(X)$ space, encodes the numerical data of

effective curves on a variety X. Its properties, such as its extremal rays, are birational invariants, guiding the contraction and flip operations in the MMP.

The MMP proceeds by a sequence of divisorial contractions and flips, which are dictated by the birational invariants. These transformations aim to achieve specific properties, such as the termination of flips or the existence of a minimal model with certain desired characteristics [50].

Another significant application of birational invariants lies in the study of the birational automorphism group of algebraic varieties. The birational automorphism group, denoted by Bir(X), captures the symmetries of a variety under birational transformations. Certain properties of this group, such as its finiteness or infiniteness, can be determined by analyzing birational invariants like the Kodaira dimension and the plurigenera [51].

9. Connections to Number Theory and Complex Geometry

The study of birational invariants has led to the development of various tools and resources that facilitate research and exploration in this field. One such platform is Project Euclid, managed by Duke University Press [52]. It provides access to a vast collection of mathematics and statistics journals, books, and proceedings, including:

- **Euclid Prime:** A collection of over 30 high-impact titles, offering term access to archival content from an additional 18 titles.
- **Advanced Studies:** Euro-Tbilisi Mathematical Journal: Relaunched in September 2021 with increased standards and broader coverage.
- **Michigan Mathematical Journal:** Published a special volume honoring Gopal Prasad, its former managing editor, on his 75th birthday.

Project Euclid offers a range of features and tools for researchers, librarians, and publishers, such as browsing, searching, subscriptions, and access management. It serves as a valuable resource for exploring the latest developments and findings in the field of birational invariants [53].

Birational maps play a crucial role in the study of birational invariants. These are rational maps between algebraic varieties that have a rational inverse, meaning the varieties are isomorphic outside of lower-dimensional subsets [54, 55]. Birational invariants are properties of algebraic varieties that remain the same or isomorphic for all varieties that are birationally equivalent. Examples include:

- Plurigenera
- Kodaira dimension
- Summands of the tensor powers of the cotangent bundle
- Hodge numbers
- Fundamental group of smooth projective varieties

Minimal models, which are the simplest varieties in a birational equivalence class, are also closely tied to birational invariants. The minimal model conjecture states that every variety is either covered by rational curves or birational to a minimal variety [56]. This conjecture highlights the significance of birational invariants in classifying and understanding algebraic varieties.

A significant recent development in the study of birational invariants is the paper "Motivic invariants of birational maps" by Hsueh-Yung Lin and Evgeny Shinder [57]. This groundbreaking work constructs invariants of birational maps with values in the Kontsevich-Tschinkel group and in the truncated Grothendieck groups of varieties. These invariants are morphisms of groupoids, making them well-suited for investigating the structure of the Grothendieck ring and L-equivalence (Fig. 4).

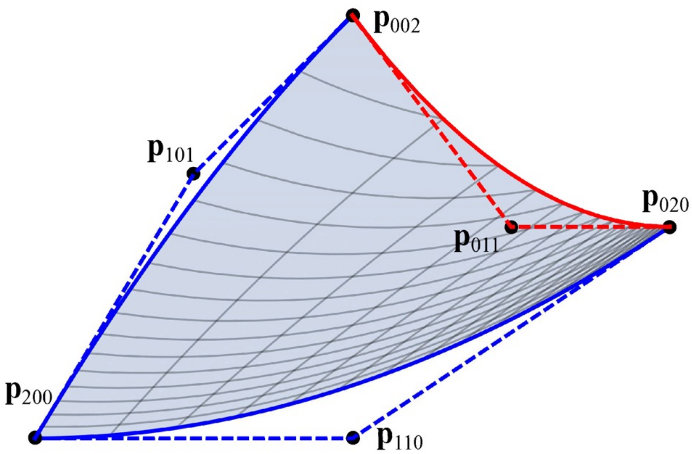

Figure 4. A birational quadratic planar map.

Building on known constructions of L-equivalence, the paper proves new and unexpected results about Cremona groups, which are groups of birational transformations of projective spaces. Some key highlights of the paper include:

- Submitted to the arXiv preprint server on July 15, 2022 (version 1), and last revised on June 11, 2023 (version 2) [58].
- Accepted for publication in the prestigious Annals of Mathematics journal.
- Categorized under Algebraic Geometry (math.AG) and K-Theory and Homology (math.KT) subjects on arXiv [59].

Another notable contribution is the paper [60, 61], which studies the growth and integrability properties of a collection of birational transformations of the complex projective space P^3. These transformations are obtained by composing the standard Cremona transformation c_3 with projectivities of finite order g in the Cremona-cubes group C. The paper identifies three possible behaviors for these maps:

1. **Integrable:** Quadratic degree growth and two invariants.
2. **Periodic:** Two-periodic degree sequences and more than two invariants.
3. **Non-integrable:** Submaximal degree growth and one invariant.

The behavior depends on the cardinality of the orbit <g>$E$of the points in E under the action of g, and the paper provides a detailed analysis of these cases. In the realm of complex dynamics, the paper [62] constructs and studies a natural invariant measure μ for a birational self-map f+ of the complex projective plane P2. Under certain conditions, such as the indeterminacy sets I+ and I- having μ-measure zero, the measure μ is shown to be (non-uniformly) hyperbolic, with its support lying in the closure of the saddle periodic points of f+. The paper also establishes the integrability of log + fhDf + fh, which is crucial for the existence of distinct Lyapunov exponents for f+.

10. Conclusion

The exploration of birational invariants has unveiled a profound and intricate realm within the field of algebraic geometry. These mathematical constructs, which remain unchanged under birational transformations, provide invaluable insights into the fundamental properties and characteristics of algebraic varieties. From the invariance of the fundamental group and Hodge numbers to the crucial roles played by the canonical divisor and Kodaira dimension, birational invariants serve as guiding beacons in the classification and study of these complex structures. The study of birational invariants has paved the way for groundbreaking developments, such as the Minimal Model Program and the construction of invariants with values in the Kontsevich-Tschinkel group and truncated Grothendieck groups. These advancements not only deepen our theoretical understanding but also open up new avenues for exploration, unlocking the potential for further discoveries and applications in various domains of mathematics and its interdisciplinary frontiers.

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